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INTERRELATION BETWEEN VARIOUS MATHEMATICAL MODELS OF THE DEFORMATION OF ELASTIC WHEELS

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Simulation of the rheological properties of elastic wheels by the Volterra integral equations of the second kind with the Koltunov kernel of the nonlinear hereditary theory of viscoelasticity and by the differential equations that under certain conditions approximately replace the adopted integral equations is suggested. The interrelationship between two different types of mathematical models of the deformation of elastic wheels is shown: without account for the time factor and with account for this factor, as well as of the rheological models: integral and differential equations. For some elastic wheels the parameters of the suggested governing equations have been found from experimental data.

In agricultural, transport, and road-building mechanical engineering, of great importance is the solution of problems of increasing the tractive-coupling properties of automobiles, tractors, and other mobile machinery, of increasing the smoothness of machine running, and of optimizing the packing of deformable bases (soils, grounds, etc.) as a result of the influence of transportation facilities. This can be favored by the application of methods that allow one to rather accurately calculate the characteristics of the processes investigated. One is to have at one's disposal formulas and algorithms of calculation of the indices of interaction of wheels equipped with pneumatic tires (elastic wheels) with deformable and with presumably undeformable (asphalt, concrete) bases. However, at the present time, the theory of the rolling motion of elastic wheels has been developed inadequately.

The accuracy of working formulas depends primarily on the choice of governing equations that model the regularities of the deformation of contacting bodies. In what follows, problems of the selection of mathematical models of deformation of elastic wheels and justification of the interrelationships between various types of these models are investigated.

Statement of the Problem. We will consider the scheme of interaction of an elastic wheel with a hard base (Fig. 1). The scheme corresponds to the longitudinal section of the wheel of radius R with its center at the point O that passes through the middle of the width B_{tir} of the tire profile. The vertical load on the wheel axis is G_0. The forces acting on the wheel are not shown. The contact surface of the wheel with the base is a segment of the straight line $B_1B = 2R \sin \psi_b$; the point C_2 located on the ordinate axis marks the middle of this segment. The normal deflection of the tire f as a result of the action of the force G_0 is comprised of the value CC_1 equal to the vertical displacement OO_1 of the wheel axis and of the maximum vertical deformation h_m of the tire at the zone of contact: $f = R - r_{\text{dyn}} = OO_1 + h_m = OO_1 + R(1 - \cos \psi_b)$.

The mathematical models of the deformation of solid bodies and structures, elastic wheels in particular, fall into two types: (1) the equations of coupling between deformations and stresses that do not include time and (2) the equations that take into account changes in time of deformations and stresses. The former describe the curves obtained as a result of stepwise static loading and subsequent stepwise unloading of bodies in shear and press tests in which only stabilized (conventionally) loadings and deformations are registered at each step of loading and deformation. The latter characterize the curves that can be obtained in various regimes of deformation with recording of loadings or stresses σ at different fixed instants of time.

Experimentally, the regularities of the deformation of elastic wheels are revealed as a result of recording loadings G and deformations h corresponding to these loadings on loading and subsequent unloading of rolling wheels in

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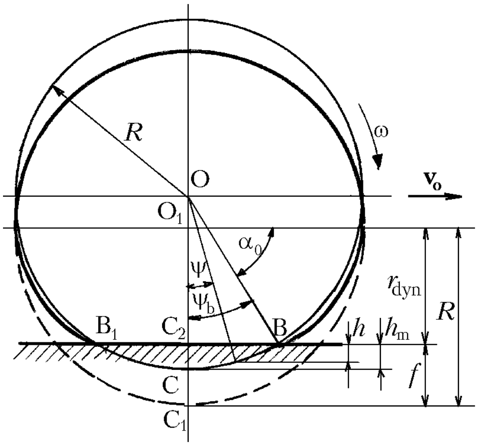

Fig. 1. Schematic diagram of the interaction of an elastic wheel with a hard base.

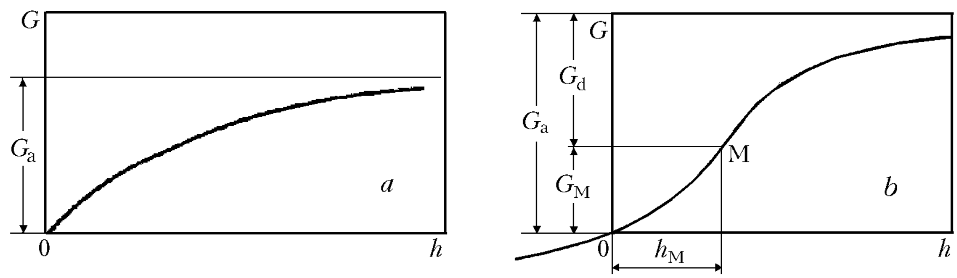

Fig. 2. Graphs of functions (2) (a) and (3) (b). G, kN; h, cm.

bench tests. The experimental relationships between h and G are generally nonlinear [1]. In order to describe the coupling $G \sim h$ one can use (just as it is done for soils in [2, 3])

the power law

$$G = kh^{\lambda}, \tag{1}$$

the equation of the hyperbolic tangent function [4] (Fig. 2a)

$$G = G_a \tanh \frac{k_1}{G_a} h, \tag{2}$$

the equation that describes the generalized coupling $G \sim h$ with the aid of the hyperbolic tangent function [5] (Fig. 2b)

$$G = G_d \left[\tanh \frac{k_2}{G_d} (h - h_M) + \tanh \frac{k_2}{G_d} h_M \right], \tag{3}$$

the third-degree polynomials without a free term [5]

$$G = q_1 h + q_2 h^2 + q_3 h^3. \tag{4}$$

In a number of cases, under real loadings acting on the tire from the side of the car, with a sufficient degree of accuracy one can adopt the linear function $G(h)$ which is described by the Hadekel formula [1]. In a transformed form it can be presented as

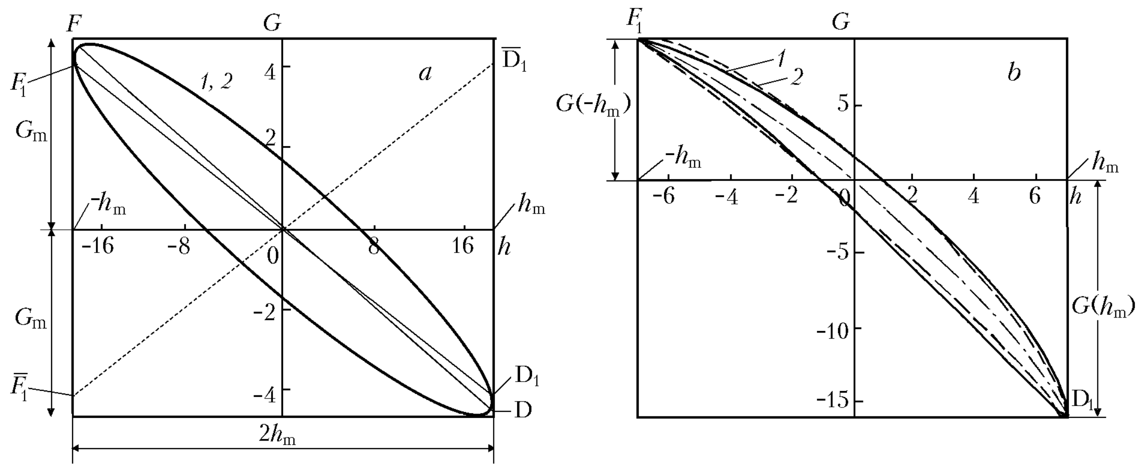

Fig. 3. Relationship between loading and deformation on cyclic radial loading of elastic wheels; a) with a tire 12.00–38 (p_{tir} = 0.14 MPa); b) with a tire 12.00–18 (p_{tir} = 0.05 MPa); 1) experiment [7, 8]; 2) calculation by Eq. (9) (1 and 2 in the scale of Fig. 3a coincide). *G*, kN; *h*, mm.

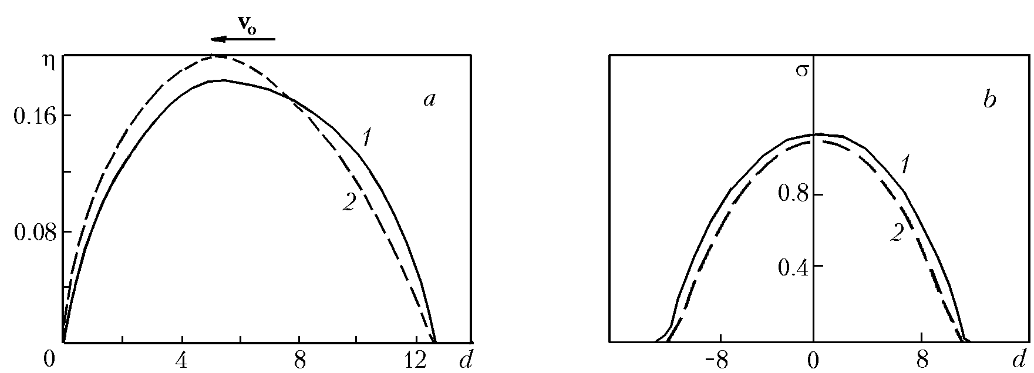

Fig. 4. Lines of regression of contact specific forces and normal stresses in rolling of elastic wheels over practically undeformable bases; a) for an automobile wheel with a tire 5.00–10 (G_0 = 1.66 kN, p_{tir} = 0.13 MPa); b) for a tractor wheel with a tire 11.2–20 (G_0 = 7.6 kN, p_{tir} = 0.26 MPa); 1 and 2) empirical and theoretical lines of regression. η, kN/cm; σ, MPa; *d*, cm.

$$G = E_{wh} h \, . \tag{5}$$

Apart from applying stepwise static loadings, constant at each stage, with subsequent stepwise unloading, another regime of deformation was implemented in a number of investigations: loading by cyclic loadings continuously changing in time *t* according to the given law $G(t)$. In both of those regimes, the diagrams of coupling between loadings and deformations represent closed hysteresis loops [6–11]. In continuous cyclic deformation, a phase shift of loadings and deformations is observed: their maxima do not coincide. Curves 1 presented in Fig. 3 describe the experimental data obtained in bench tests.

The regularities of deformation of elastic wheels can also be revealed from experimental diagrams that characterize the distribution of specific forces $\eta(t) = (G_0/B_{tir})(t)$ or normal stresses $\sigma(t)$ that appear in rolling of a wheel over a hard base [9, 12–14]. Curves 1 presented in Fig. 4 describe the experimental data obtained for rolling wheels. The diagrams correspond to the longitudinal section of a wheel that passes through the middle of the tire profile width: $d = R \sin \psi$. The diagrams of $\eta(t)$ and $\sigma(t)$ reveal the same properties of elastic wheels as do the diagrams of $G(h)$ plotted by the results of bench tests. These analogies are especially evident if, excluding time, we transform the

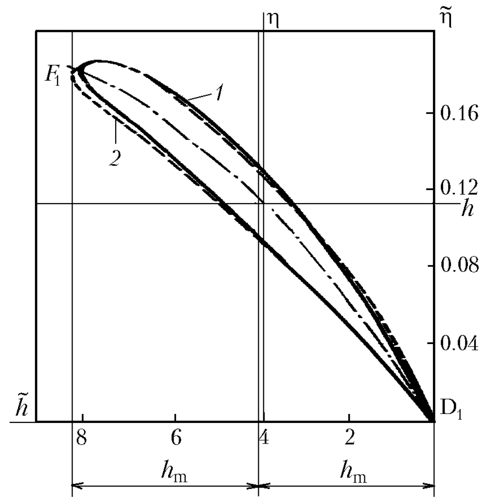

Fig. 5. Relationship between specific forces and deformations for an automobile wheel with a tire 5.00–10: 1) experimental curve corresponding to curve 1 in Fig. 4a; 2) theoretical curve according an equation of the form of (11). η, kN/cm; *h*, cm.

diagrams of $\eta(t)$ into the $\eta(h)$ curves (or $\sigma(t)$ into $\sigma(h)$). The $\eta(h)$ and $\sigma(h)$ curves represent the closed hysteresis loops. Curve 1 given in Fig. 5 is the experimental $\eta(h)$ curve obtained by transformation of the $\eta(t)$ plot.

Hysteresis loops are formed in all of the regimes of loading and subsequent unloading of elastic wheels; they reveal the viscoelastic properties of wheels. These properties are also revealed in investigations of the change in time in the amplitudes of free and forced vibrations of wheels [15, 16]. However, the governing equations (1)–(5) (of the first type) used have substantial drawbacks. They do not allow one to take into account the phase shift of stresses and deformations, to justify the formation of hysteresis loops on loadings and subsequent unloadings of elastic wheels, and to develop the method of calculation of the deformation time- and rate-dependent indices of the interaction of wheels with deformed bases. The deformations of wheels with $\sigma < \sigma_{str}$ are entirely reversible, but the relations of the first type do not reflect this.

The indicated drawbacks are justifiably eliminated by using the governing relations of the second type — rheological equations (the governing equations of viscoelasticity theory). They describe the curves that can be obtained in different regimes and conditions of continuous deformation with recording of stresses and deformations at different fixed instants of time.

The viscoelastic properties of elastic wheels have a number of characteristic features. It has been revealed experimentally that: 1) the areas of the hysteresis loops and the numerical values of $G(h)$ and $\sigma(h)$ on these diagrams are practically independent of the regime of continuous deformation [8, 16]; 2) the areas of the hysteresis loops formed during forced vibrations of elastic wheels with a constant amplitude of deformations and with different frequencies of vibrations practically do not depend on the frequency of vibrations [8, 16]; 3) the rolling speed (up to 80–100 km/h) does not influence the values and character of distribution of stresses $\sigma(t)$ when a tire is in contact with a hard base [9, 14, 17].

These experimental data are perceived as contradicting the inferences of the theories of viscoelasticity and of attenuating vibrations of a medium in which the internal (viscous) resistance of the medium is taken proportional to the deformation rate. Therefore, the mathematical simulation of the regularities of the deformation of elastic wheels is encountered with difficulty.

In a number of works, the regularities of the deformation of elastic wheels are modeled by differential equations of viscoelasticity theory. The simplest of these are the equations that describe the properties of the Kelvin and Maxwell ideal mechanical models [18], respectively:

$$\sigma = E\varepsilon + \mu\varepsilon'_t, \tag{6}$$

$$\sigma'_t + \frac{1}{T}\sigma = E\varepsilon'_t , \qquad (7)$$

where $T = \mu/E$. However, the use of relations (6) and (7) suffers from substantial drawbacks. In particular, Eq. (6) does not allow one to describe the experimentally revealed relaxation of stresses in tires as well as the diagram of $\sigma(t)$. For the Maxwell model the viscous part of the deformation is residual, whereas the deformation of elastic wheels on unloading is reversible.

No differential equations with constant coefficients that describe the properties of ideal deformed media may satisfactorily reflect all the features of the viscoelastic properties of elastic wheels. Rheological models in the form of differential equations do not generalize the results of investigations represented by Eqs. (1)–(5) that do not include time but are perceived as independent of them.

The Boltzmann–Volterra hereditary theory [19–23] is the most general one that allows simulation of the processes of deformation of viscoelastic media and structures. We will use the apparatus of the integral equations of this theory to model the regularities of the deformation of elastic wheels in time.

We designate the radial force and the radial deformation of a wheel by G and h. We will describe the regularity of radial deformation of elastic wheels in time by the Volterra linear integral equation of the second kind:

$$\Phi\,[h\,(t)] = G\,(t) + \int_{-\infty}^{t} K\,(t-\tau)\,G\,(\tau)\,d\tau\,. \qquad (8)$$

Its solution has the form

$$G\,(t) = \Phi\,[h\,(t)] - \int_{-\infty}^{t} T\,(t-\tau)\,\Phi\,[h\,(\tau)]\,d\tau\,. \qquad (9)$$

In modeling the regularity of tangential deformation of an elastic wheel we will use equations similar to Eqs. (8) and (9), where we replace $G(t)$ and $h(t)$ by $F(t)$ and $s(t)$, respectively.

The free term $\Phi[h(t)]$ in (8) is generally the nonlinear function of h. The function $\Phi(h)$ describes the line $G_{\mathrm{dir}} = \Phi(h)$ which will be called by us the directrix of the hysteresis loop. The interrelated functions $K(t)$ and $T(t)$, which are the kernel and resolvent of Eq. (8), also called the functions of the creep and relaxation rates, allow one to take into account the influence of the previous stressed state of the tire on its stressed state at the given instant of time.

In modeling the coupling between stresses σ and deformations h in the case of the rolling of wheels the following equations will correspond to Eqs. (8) and (9):

$$\Phi\,[h\,(t)] = \sigma\,(t) + \int_{-\infty}^{t} K\,(t-\tau)\,\sigma\,(\tau)\,d\tau\,, \qquad (10)$$

$$\sigma\,(t) = \Phi\,[h\,(t)] - \int_{-\infty}^{t} T\,(t-\tau)\,\Phi\,[h\,(\tau)]\,d\tau\,. \qquad (11)$$

To obtain a rather accurate description of experimental data by Eqs. (8) and (10) it is necessary that the functions $K(t)$ and $T(t)$ could satisfy certain conditions [21, 22]. As functions of the creep and relaxation rates we have taken the following functions that satisfy these conditions:

$$K\,(t) = \frac{\exp\,(-\beta t)}{t} \sum_{n=1}^{\infty} \frac{[A\Gamma\,(\alpha)]^n\, t^{\alpha n}}{\Gamma\,(\alpha n)}\,, \quad 0 < \alpha < 1\,; \qquad (12)$$

$$T(t) = A \exp(-\beta t)\, t^{\alpha-1} \tag{13}$$

(the Koltunov kernel and the Rzhanitsyn kernel, respectively). Having replaced the independent variable τ in (9) by $\Theta = t - \tau$, we obtain

$$G(t) = \Phi[h(t)] - \int_0^\infty T(\Theta)\, \Phi[h(t-\Theta)]\, d\Theta\,. \tag{14}$$

The laws of viscoelastic deformation of elastic wheels are subdivided into linear and nonlinear. The experimental hysteresis loops obtained in radial or tangential continuous deformation of elastic wheels in the region of linearity of their properties have the shape of an ellipse [7, 12, 23] (Fig. 3a), whereas in the region of nonlinearity of properties they have a different shape [8, 9] (curves 1 in Fig. 3b and Fig. 5). The linearity or nonlinearity of the properties of an elastic wheel with a tire of the given construction depends on the amplitude of loading G_m (or the amplitude of deformation h_m) and the internal pressure of air in the tire p_{tir}.

Let an elastic wheel possess linear properties. In this case, the directrix of the diagrams of $G = G(h)$ and $\sigma = \sigma(h)$ is a straight line, $\Phi(h) = qh$. Then Eq. (14) will acquire the form

$$G(t) = q\left[h(t) - \int_0^\infty T(\Theta)\, h(t-\Theta)\, d\Theta\right]. \tag{15}$$

In describing the dependence $\sigma = \sigma(h)$, the quantity $G(t)$ on the left-hand side of Eq. (15) is replaced by $\sigma(t)$.

We will consider a steady regime of the deformation of a wheel according to the sinusoidal law:

$$h(t) = h_m \sin \omega t\,. \tag{16}$$

In this case, from (15) it follows that

$$G(t) = qh_m\,[A(\omega)\cos\omega t + (1 - B(\omega))\sin\omega t]\,, \tag{17}$$

where

$$A(\omega) = \int_0^\infty T(\Theta)\sin\omega\Theta\, d\Theta\,; \quad B(\omega) = \int_0^\infty T(\Theta)\cos\omega\Theta\, d\Theta\,. \tag{18}$$

For function (13) we have

$$A(\omega) = A\Gamma(\alpha)\sin(\alpha\varphi_1)/\rho^\alpha\,, \quad B(\omega) = A\Gamma(\alpha)\cos(\alpha\varphi_1)/\rho^\alpha\,, \tag{19}$$

where $\rho = \sqrt{\beta^2 + \omega^2}$; $\tan\varphi_1 = \omega/\beta$ [19]. The parameters of the function (13) can be selected so that Eq. (15) will take the form

$$G(t) = G_m \sin(\omega t + \varphi_{sh})\,, \tag{20}$$

where $G_m = E_{wh} h_m$.

To determine the parameters of the function (13), we will find the values of $A(\omega)$ and $B(\omega)$ from experimental data with the aid of the formulas that follow from Eq. (17):

$$A(\omega) = G(0)/(qh_m)\,, \quad B(\omega) = 1 - G(\pi/2\omega)/(qh_m)\,. \tag{21}$$

TABLE 1. Characteristics of the Viscoelastic Properties of Elastic Wheels (according to the results of bench tests on cyclic deformation, by loadings varying in time)

Properties of deformations	Source of information	Tires	p_{tir}, MPa	Form of loading	h_m, cm	Parameters of Eq. (9)			
						A	α	a_1, kN/cm	a_2, kN/cm^2
Linear	[7]	12.00—38	0.14	Radial	1.88	0.0122	0.006	2.696	0
	[7]	12.00—38	0.08	Same	0.64	0.0309	0.015	2.585	0
	[9]	5.00—10	0.13	Tangential at G_{rad} = 1.66 kN	0.22	0.0204	0.010	3.574	0
Nonlinear	[11]	9.00—20	0.3	Radial	4.9	0.0101	0.005	3.795	0.094
	[8]	12.00—18	0.05	Same	6.75	0.0227	0.011	0.889	0.0421
	[9]	6.40—13	0.2	Tangential at G_{rad} = 0.77 kN	0.28	0.0312	0.015	4.400	–1.441

TABLE 2. Characteristics of the Viscoelastic Properties of Elastic Wheels with a Width of 9.00–20 (according to the results of investigation of the rolling of wheels over a hard base)

Source of experimental data	p_{tir}, MPa	G, kN	h_m, mm	Parameters of Eq. (11)				
				A	α	q_1, MPa/mm	q_2, MPa/mm^2	q_2, MPa/mm^3
[13]	0.45	15.5	7.0	0.0060	0.003	0.0410	0.0058	0.0006
[12]	0.53	18.6	7.0	0.0101	0.005	0.0256	0.0043	0.0004

Having excluded the time t from Eqs. (16) and (20), we obtain the equation of an ellipse. It can also be obtained if one proceeds from the sinusoidal law of a change in loading: $G(t) = G_m \sin \omega t$. This shows that Eq. (15) with the influence function (13) exactly describes the regularities in the deformation of elastic wheels by the harmonic law in the region of linearity of their properties.

In order to find $A(\omega)$ and $B(\omega)$ from Eq. (21) we will consider the diagrams of $G(h)$ in a rectangular coordinate system hG, the axis G of which passes through the middle of the segment of length $2h_m$ equal to the doubled amplitude of the deformation of the "loading–unloading" cycle (Fig. 3a). The directions of the coordinate axes and of the tracing of the hysteresis loop are selected such that $\pi/2 < \varphi_{sh} < \pi$ should hold. In this coordinate system the directrix D_1F_1 is symmetrical with the ellipse diameter $G = G(h)$ that passes through the points $D_1(h_m, G(h_m))$ and $F_1(-h_m, G(-h_m))$.

The shift of the phases of loadings and deformations is taken into account with the aid of the function $T(t-\tau) > 0$. The larger the eccentricity of the ellipse, i.e., the smaller the value of φ_{sh}, the smaller values the function $T(t-\tau)$ has at the same t. When $\varphi_{sh} \to 0$, we obtain $G(0) \to 0$, $G(\pi/2\omega) \to G_m$, $q \to E_{wh}$. If we adopt conventionally that $T(t-\tau) = 0$ at any t, then $A(\omega) = B(\omega) = 0$, $\varphi_{sh} = 0$, and the directrix will occupy the position of the diagonal DF of the rectangle with sides $2h_m$ and $2G_m$, with the ellipse being inscribed into this rectangle (Fig. 3a). At $T(t-\tau) = 0$ the deformations will be elastic and the ellipse will degenerate into the DF segment of the straight line. We will call it the theoretical straight line of linear elasticity. For a linearly resilient elastic wheel dependence (5) holds.

Let an elastic wheel possesses nonlinear viscoelastic properties. In this case, the directrix of the hysteresis loop $G(h)$ takes the form of a curve (Fig. 3b, line F_10D_1). When $\varphi_{sh} \to 0$, it tends to the position of the theoretical curve of nonlinear elasticity. At loadings smaller than the limit of the carrying ability of the tire, the theoretical curves of the nonlinear elasticity and the directing curves corresponding to them $G_{dir} = \Phi(h)$ can be concave and convex (Fig. 2a, Fig. 3b), and they can also have concave and convex portions (Fig. 2b). Our theoretical investigations and calculations have shown that generally the curves $G_{dir} = \Phi(h)$ may be characterized by equations of the form (3) (in this case, in Fig. 2 we have $G = G_{dir}$). The convex directing curves are described by an equation of the form of Eq. (2). An equation of the form of (3) of the curve $G_{dir} = \Phi(h)$ can be approximated very accurately by polynomial (4).

Results of Calculations. We have developed a technique for determining the parameters of Eqs. (9) and (11) with kernel (13) that model the viscoelastic properties of elastic wheels by means of processing experimental closed hysteresis loops obtained on deformation of wheels according to the law (16). By the proposed technique we have found the characteristics of the viscoelastic properties of a number of automobile and tractor wheels (Tables 1 and 2). For the experimental data presented in Table 1, in the equation of the directrix (4) the parameter $q_3 = 0$. It has been found that according to the experimental data at the obtained values of the parameters A, α, β, q_1, q_2, and q_3 the calculated values of $G(t)$ and of the areas of hysteresis loops are virtually independent of the regime of continuous cyclic deformation in a harmonic manner. The values of $G(t)$ (curves 2 in Fig. 3b and Fig. 5) calculated by us with a high degree of accuracy coincide with the corresponding experimental values (curves 1 in the indicated figures), with practically complete coincidence for linearly deformed wheels (see Fig. 3a).

We assume that $T(t-\tau) = p \exp(-p(t-\tau))$; then $K(t-\tau) = p$. In the case of deformation of an elastic wheel in a harmonic manner with frequency ω, we assume that $p = \omega g$, where g is the dimensionless parameter independent of ω. Let $\Phi(h) = qh$. Having differentiated Eq. (8) under these conditions with respect to time t, we obtain

$$G'_t + \omega g G = q h'_t . \tag{22}$$

The solution of Eq. (22) that corresponds to deformation by law (16) exactly describes $G(t)$ in the case of the linearity of the properties of an elastic wheel. At q = const and g = const, the calculated area of the hysteresis loop that corresponds to the modeling of the properties of an elastic wheel by Eq. (22) with parameters q and g is independent of ω, in agreement with experimental data. We have found that

$$q = G_{\mathrm{m}} / \left(h_{\mathrm{m}} \left| \cos \varphi_{\mathrm{sh}} \right| \right), \quad g = \left| \tan \varphi_{\mathrm{sh}} \right| . \tag{23}$$

If $\varphi_{\mathrm{sh}} \to 0$, then $g \to 0$, $q \to G_{\mathrm{m}}/h_{\mathrm{m}} = E_{\mathrm{wh}}$. Thus, g characterizes the viscous properties of an elastic wheel and q — its elastic and viscous properties. We have developed a technique for determining the characteristics g and q of viscoelastic properties of an elastic wheel. Using this technique, the values of g and q have been found for certain automobile and tractor wheels.

Calculations have shown that Eq. (22) can be used to approximately describe the regularities of the deformation of viscoelastic wheels that have nonlinear properties. The higher the rigidity of a wheel and smaller the vertical loading on its axis, the smaller the errors in the determination of $G(t)$ on replacing Eq. (8) by Eq. (22).

In order to solve certain practical problems we should consider the regimes of deformation of wheels in which forces or stresses at each instant of time are to be found with consideration for the boundary-value conditions. We obtain the governing equation that corresponds to this requirement and that models the couplings between stresses σ and deformations h in rolling of wheels. We will differentiate (22) with respect to t, and in the equation obtained we replace the loading G by the stress σ and the parameter q by the parameter $\bar{q}$. This will yield

$$\sigma''_{tt} + \omega g \sigma'_t = \bar{q} h''_{tt} . \tag{24}$$

Let, in rolling of a wheel, the curves $\sigma(t)$ be obtained. We use Eq. (24) to describe the function $\sigma(t)$ that satisfies the boundary-value condition $\sigma(0) = 0$ and $\sigma(t_{\mathrm{int}}) = 0$. In accordance with the scheme of interaction of a rolling elastic wheel with a nondeformed base (Fig. 1),

$$h = R \left(\sin(\alpha_0 + \omega t) - \sin \alpha_0 \right) , \tag{25}$$

where $\alpha_0 = \pi/2 - \psi_{\mathrm{b}}$; $t \in [0, t_{\mathrm{int}}]$. Having replaced t in Eq. (25) by the new variable $\psi = \psi_{\mathrm{b}} - \omega t$, we obtain the general solution of Eq. (24) in the form

$$\sigma(\psi) = \bar{q} R \left(\tilde{C}_1 + \tilde{C}_2 \exp(-g(\psi_{\mathrm{b}} - \psi)) + \cos \psi + g \sin \psi \right) / \left(g^2 + 1 \right) . \tag{26}$$

We find the arbitrary constants $\tilde{C}_1$ and $\tilde{C}_2$ from the boundary-value conditions $\sigma(\psi_{\mathrm{b}}) = 0$ and $\sigma(-\psi_{\mathrm{b}}) = 0$.

By applying Eq. (26) we may find the parameters of Eq. (24) by processing the diagrams of normal contact stresses recorded for elastic wheels rolling over hard bases. We have developed the corresponding procedure of determining the values of g and $\bar{q}$ and by it these values have been found for some elastic wheels. The values of σ obtained from Eq. (26) are independent of the rolling speed. The theoretical plots constructed from the results of calculations by this equation with the found values of the characteristics g and $\bar{q}$ rather accurately approximate experimental diagrams (Fig. 4a).

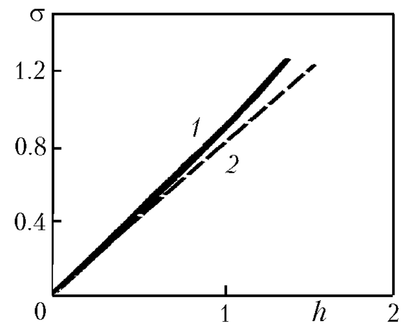

Fig. 6. Relationship between normal contact stresses and deformations for a tractor wheel with a tire 11.2–20: 1) the straight line corresponding to experimental curve 1 in Fig. 4b; 2) the straight line corresponding to Eq. (27). σ, MPa; h, cm.

We have carried out Fisher-criterion checking at a 5%-significance level of the adequacy of describing the stress σ by Eq. (24) in the region of the nonlinearity of the viscoelastic properties of elastic wheels. It has proved the validity of Eq. (24) for modeling the regularity of the change of σ for many elastic wheels in this region. At the same time, it was shown that in the region of the nonlinearity of the properties of elastic wheels integral equations (8) and (10) model these properties more accurately than differential equations (22) and (24).

Thus, the results obtained demonstrate the interrelationship of two types of rheological models for elastic wheels: integral and differential equations.

At $\psi_{sh} = 0$, the diagram of $\sigma(\psi)$ becomes symmetrical with the maximum that corresponds to $\psi = 0$ and that of $\sigma(d)$ — symmetrical with the maximum that corresponds to $d = 0$ (curve 1 in Fig. 4b). The loading and unloading branches on the hysteresis loops virtually coincide, i.e., elastic wheels undergo deformation as virtually resilient ones (straight line 1 in Fig. 6). The regularity of the deformation of linearly elastic wheels is described by Eq. (5) or for $\sigma = \sigma(h)$ by

$$\sigma = \overline{E}_{wh} h \,. \tag{27}$$

We have worked out the calculation method of determining the coefficient $\overline{E}_{wh}$ and, using it, we have found the coefficients $\overline{E}_{wh}$ of some elastic wheels depending on G_0, p_{tir}, and the construction parameters of wheels. The value of $\overline{E}_{wh}$ in theoretical relation (27) represented by straight line 2 in Fig. 6 was calculated by the method suggested by us. The mean relative deviation and mean standard deviation of the calculated values of $\overline{E}_{wh}$ from the corresponding experimental ones for wheels with tires 11.2–20 are equal to 7.12 and 4.52%. The discrepancies do not exceed experimental errors.

Let, in bench tests, there be stepwise static loading of an elastic wheel by radial loadings constant at each stage. In this case, in accordance with the available experimental data the relationship between the deformations and loadings varying in time will be described in the region of nonlinearity of the properties of an elastic wheel by the integral equation

$$\Phi_0 (h) = G (t) + \int_0^t K (t - \tau) \, G (\tau) \, d\tau \tag{28}$$

with kernel (12). In (28), $G = \Phi_0(h)$ is the equation of the curve of instantaneous deformation or of the isochronous curve $G = G(h)$ at $t = 0$ [21].

Assuming in (28) that $G(t) = G_i = \text{const}$, we obtain the equations of the creep curves:

$$\Phi_0 (h) = G_i \tilde{f}(t) \,, \quad i = 1, 2, ..., r \,, \tag{29}$$

where

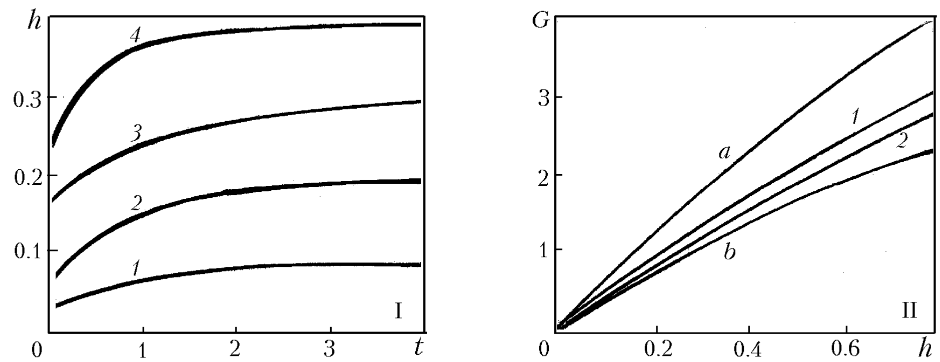

Fig. 7. Creep on radial loading of a resilient viscoelastic wheel (h_m = 0.4 cm): I) creep curves [1) G = 0.25; 2) 0.5; 3) 1.0; 4) 1.5]; II) isochronic curves [1) t = 0.2; 2) 1.0]; a and b) calculated curves $G = \Phi_0(h)$ and $G = \Phi_\infty(h)$, respectively. G, kN; h, cm; t, sec.

$$\tilde{f}(t) = 1 + \int_0^t K(\tau)\, d\tau \, . \tag{30}$$

Relation (29) reflects the fact of the similarity of isochronous curves, i.e., of the curves of the relations $G = \Phi_j(h)$ at different fixed instants of time $t = t_j$ = const (j = 1, 2, ...). Figure 7I presents a family of creep curves $h(t)$ at different values of G_i and Fig. 7II the corresponding graphs of isochronic curves $G = \Phi_j(h)$ at different values of t of the elastic wheel the properties of which are modeled by Eq. (28) with kernel (12). The characteristics of the viscoelastic properties of the wheel are: α = 0.025, β = 0.05, A = 0.0176, q_1 = 6.5 kN/cm, q_2 = −1.9 kN/cm^2, and q_3 = 0. In constructing the graphs the numerical values of function (30) given in [21] were used.

If, during the action of the force G_i, the stresses in a tire do not exceed its limit of carrying ability, limited creep appears. In this case, it is possible, by using the theoretical value of function (30) at t corresponding to the time of conventional stabilization of deformation, to find the curve of the limiting state $G = \Phi_\infty(h)$, that is, an isochrone with the mark $t = \infty$. The curves $G = \Phi_0(h)$ and $G = \Phi_\infty(h)$ limit, respectively from above and from below, the entire possible region of deformation of the elastic wheel. The lines $G = \Phi_0(h)$ and $G = \Phi_\infty(h)$ that correspond to the same family of creep curves are similar and can be approximated by equations of the same form. These are Eqs. (2)–(4). In the region of linearity of the properties of a viscoelastic wheel the graphs of the functions $G = \Phi_0(h)$ and $G = \Phi_\infty(h)$ are straight lines the equations of which have the form $\Phi_0(h) = (E_{wh})_0 h$ and $\Phi_\infty(h) = (E_{wh})_\infty h$. If a wheel is deformed as a linearly elastic one, the graphs of the functions $G = \Phi_0(h)$ and $G = \Phi_\infty(h)$ coincide; the dependence $G = G(h)$ is described by Eq. (5).

The application of Eq. (28) makes it possible to supplement and generalize the results of investigations on mathematical simulation of the regularities of the deformation of elastic wheels without allowance for the factor of time, to reveal the interrelationship between two different types of mathematical models of deformation of elastic wheels — without allowance for the factor of time and with its allowance. Indeed, on the basis of Eq. (29), any of the governing equations of the first type can be considered as the equation of an isochrone with the mark $t = t_{st}$, which is the time of conventional stabilization of deformation at each stage of unloading in static tests, without allowance for the factor of time. Mathematical models without allowance for the factor of time are transferred into rheological equations by using the values of function (30).

The proposed mathematical models of deformation of elastic wheels can be used as governing equations in calculations for selecting optimal construction parameters of the running gears and regimes of operation of mobile power facilities.

CONCLUSIONS

1. Mathematical models of the deformation of elastic wheels under different conditions of their loading are suggested and justified: integral equations (8), (10), (28) with kernel (12) and differential equations (22) and (24). In solving the problems of interaction of mobile machines with various bearing bases, Eq. (24) is recommended as the governing one for elastic wheels.

2. The regularity of radial deformation of linearly resilient elastic wheels is modeled in determining the functions $G(h)$ and $\sigma(h)$ by Eqs. (5) and (27), respectively.

3. The methods of determining the characteristics of viscoelastic and elastic properties of elastic wheels from experimental data have been developed and implemented for a number of automobile and tractor wheels.

4. Based on the use of the proposed governing integral equations, the interrelationship between two different types of mathematical models of deformation of elastic wheels — without account for the factor of time and with its account — has been revealed, as well as the interrelationship of two rheological models: integral and differential equations. This allows one to justifiably determine the conditions and limits of applicability of each of the equations in solving practical problems of the selection of optimal construction parameters and regimes of operation of mobile machines.

NOTATION

A, α, β, characteristics of the viscoelastic properties of an elastic wheel (the parameters of the Koltunov kernel and Rzhanitsyn kernel) determined experimentally; $A(\omega)$ and $B(\omega)$, sine and cosine of the transformation of the function $T(\Theta)$ in Eq. (15); B_{tir}, width of the tire profile, m; $\tilde{C}_1$ and $\tilde{C}_2$, arbitrary constants in Eq. (26); d, abscissas of the points on the line of contact of an elastic wheel with a hard base, cm; E, elasticity modulus of a medium or material, MPa; E_{wh}, elasticity coefficient of an elastic wheel in Eq. (5), kN/cm; $\bar{E}_{\text{wh}}$, elasticity coefficient of an elastic wheel in Eq. (27), MN/m^3; $(E_{\text{wh}})_0$ and $(E_{\text{wh}})_\infty$, instantaneous and limiting coefficients of the elasticity of a viscoelastic elastic wheel, kN/cm; $F(t)$, shearing force in tangential deformation of an elastic wheel, kN; f, normal tire deflection, m; $\tilde{f}(t)$, flexibility function; G_0, vertical loading on the wheel axis in its rolling over hard base, kN; G_{a}, limit of the carrying ability of a tire, kN; $G_{\text{d}} = G_{\text{a}} - G_{\text{M}}$, difference between the carrying ability of a tire and loading G_{M} at the inflection point M in the graph of function (3), kN; G_{dir}, loading corresponding to the points of the directrix of the hysteresis loop, kN; G_{M}, loading corresponding to the points of the inflection points of the curve described by Eq. (3), kN; G_{m}, amplitude of loading, kN; $G(t)$, vertical loading on the wheel axis during its radial deformation in bench tests, kN; $G'(t)$, first derivative of $G(t)$ in time; G_{rad}, radial loading on the wheel axis on its tangential deformation, kN; g, characteristic of the viscous properties of an elastic wheel; h, current value of the compression deformation of a tire in the zone of contact with the base, cm; h_{m}, maximum deformation of a tire at the zone of its contact with a hard base (deformation amplitude), cm; h_{M}, abscissa of the inflection point M in the graph of function (3), cm; h'_t, first derivative of $h(t)$ in time; h''_{tt}, second derivative of $h(t)$ in time; $(\tilde{h}, \tilde{\eta})$, coordinates of the points of the $\eta(\tilde{h})$ curves in the coordinate system $\tilde{h}D_1\tilde{\eta}$ (Fig. 5), cm, kN/cm; $K(t)$, kernel of integral equation (8); k, empirical coefficient in Eq. (1); k_1, angular coefficient of the tangent drawn to the graph of function (2) at the coordinate origin; k_2, angular coefficient of the tangent to the graph of function (3) at the inflection point (h_{M}, G_{M}) (Fig. 2b); p, parameter in the function $T(t-\tau)$, cm^{-1}; p_{tir}, internal pressure of air in the tire, MPa; q, angular coefficient of the directrix of a straight hysteresis loop of a viscoelastic resilient wheel in the region of the linearity of its properties, the parameter of the governing differential equation (22), kN/cm; $\bar{q}$, parameter of the governing differential equation (24) for an elastic wheel (the characteristic of the viscoelastic properties of an elastic wheel, MN/m^3; q_1, q_2, q_3, coefficients in Eq. (4), kN/cm, kN/cm^2, kN/cm^3; R, radius of the outside circle of the tire, m; r_{dyn}, dynamic radius of an elastic wheel (in rolling) or static radius of an elastic wheel (on deformation of an unrolling wheel), m; s, shear strain on tangential deformation of an elastic wheel, cm; T, period of relaxation of a medium or material, sec; $T(t)$, resolvent of the integral equation (8); t, time, sec; t_{int}, time of interaction of a wheel with a base on one rotation of the wheel around its axis, sec; t_{st}, time of conventional stabilization of deformation at each stage of loading in static tests without account for the factor of time, sec; v_{ax}, velocity of the wheel axis, m/sec; α_0, angle characterizing the position of the point B at which the wheel enters into contact with the base (Fig. 1 and Eq. (25)), rad; $\Gamma(\alpha)$, gamma-function; ε, relative deformation of compression; η, specific forces (per unit width of the tire profile), kN/cm; Θ, integration variable; λ, exponent in Eq. (1) (empirical coefficient); μ, viscosity coefficient, MPa·sec; ρ, parameter in Eqs. (19); φ_1, parameter in Eqs. (19); σ, compressive stress, MPa; σ'_t, first derivative of σ(*t*) in time; σ''_{tt}, second derivative of σ(*t*) in time; σ_{str}, ultimate strength, MPa; σ(ψ), stress of tire compression at the points of the line of contact of an elastic wheel, MPa; τ, current value of time in Eqs. (8) and (9) that precedes the time moment *t*, sec; Φ[*h*(*t*)], free term in Eqs. (8) and (9); $\Phi_0(h)$, curve of instantaneous deformation (isochrone with mark $t = 0$); $\Phi_\infty(h)$, curve that characterizes stabilized deformations of a tire (isochrone with mark $t = \infty$); φ_{sh}, angle of deformation phase shift relative to the loading phase, rad; ψ, current value of the angle of contact of a tire with a base, deg; ψ_b, angle equal to half the central angle of the circle of radius *R* that corresponds to the chord B_1B (Fig. 1) and respectively equal to the maximum angle contact of a wheel in Eq. (26), deg; ω, frequency of the harmonic process of deformation (angular velocity of a rolling wheel), sec^{-1}. Subscripts and superscripts: a, asymptote; ax, wheel axis; b, half the angle of contact of an elastic wheel with a hard base (corresponding to point B in Fig. 1); d, difference; dir, directrix; dyn, dynamic; int, interaction; *i*, number of the creep curve; *j*, number of isochronic curve corresponding to a fixed moment of time t_j; m, maximum; *n*, ordinal number of the series term in Eq. (12); *r*, number of creep curves in the family ($i = 1, 2, ..., r$); rad, radial (loading); sh, shift (of loading phases and deformation); st, stabilization; str, strength; *t*, first derivative in time; *tt*, second derivative in time; tir, tire; wh, wheel; ′ and ″, first and second derivatives of a function.

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